On Tautological Globular Operads

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(Non-symmetric) Operads Classically Defined

Definition

A **non-symmetric operad** O consists of a sequence of sets $\{O(n)\}_{n\in\mathbb{N}}$ whose n-th entry is the set of n-ary operations, an identity operation $\mathbb{1} \in O(1)$, and for all $n, k_1, k_2, ..., k_n \in \mathbb{N}$ a composition operation

$$\circ : O(n) \times \prod_{i=1}^{n} O(k_i) \to O(\sum_{i=1}^{n} k_i)$$

such that

$$\theta_0 \circ (\theta_1 \circ (\theta_{1_1}, ..., \theta_{1_k}), ..., \theta_n \circ (\theta_{n_1}, ..., \theta_{n_l})) =$$

$$(\theta_0 \circ (\theta_1, ..., \theta_n)) \circ (\theta_{1_1}, ..., \theta_{1_k}, ..., \theta_{n_1}, ..., \theta_{n_l})$$

and

$$\theta_0 \circ (\mathbb{1}, \mathbb{1}, ..., \mathbb{1}) = \theta_0 = \mathbb{1} \circ \theta_0$$

for all $\theta_i \in \{O(n)\}_{n\in\mathbb{N}}$ whenever the compositions are well-defined.

Algebra for an Operad

Definition

The **tautological operad** $taut(X)$ on a set X is the operad whose

- n-ary operations are $\mathbf{Set}(X^n, X)$
- Identity operation is the identity map $\mathbb{1}_X : X \to X$
- Composition is given by

$$\circ : \mathbf{Set}(X^n, X) \times \prod_{i=1}^{n} \mathbf{Set}(X^{k_i}, X) \to \mathbf{Set}(X^n, X) \times \mathbf{Set}(X^{\sum_i k_i}, X^n)$$

$$\to \mathbf{Set}(X^{\sum_i k_i}, X)$$

Definition

An **algebra** A for an operad O is a set A equipped with an operad homomorphism $\xi : O \to taut(A)$.

The Category of Graded Sets

- The category **GrdSet** is the slice category $\mathbf{Set}/\mathbb{N}$

The Free Monoid Monad T

$T : \mathbf{Set} \to \mathbf{Set}$

- $\mu : T^2 \Rightarrow T$
- $\eta : \mathbb{1}_{\mathbf{Set}} \Rightarrow T$

We will freely interchange the set $\mathbb{N}$ with $T(\{*\})$

The Composition Tensor Product

Definition

Let $x : X \to \mathbb{N}$ and $y : Y \to \mathbb{N}$ be a pair of graded sets. Their **composition tensor product** $x \Box y : X \Box Y \to \mathbb{N}$ is defined by the diagram

$$\begin{tikzcd}
X \Box Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & T(Y) \arrow[r, "T(y)"] \arrow[d, "T(!_Y)"] & T^2(\{*\}) \arrow[r, "\mu_{\{*\}}"] & T(\{*\}) \\
X \arrow[r, "x"] & T(\{*\}) & &
\end{tikzcd}$$

where $!_Y : Y \to \{*\}$ is the unique map from Y to the terminal one point set. The underlying graded set $X \Box Y$ is the pullback of x and $T(!_Y)$ with the arity function $x \Box y$ defined to be the composition along the top row.

Operads as Monoids

The product $\square$ together with the graded set $i : \{*\} \hookrightarrow T(\{*\})$ gives **GrdSet** the structure of a monoidal category.

Definition

A **nonsymmetric operad** is a monoid $(X, m : X \square X \to X, e : \{*\} \to X)$ in **GrdSet** with respect to the composition tensor product $\square$.

$$\begin{tikzcd}
(X\square X)\square X \arrow[r, "\alpha^{\mathbf{GrdSet}}_{X,X,X}"] \arrow[d, "m\square\mathbb{1}_X"'] & X\square(X\square X) \arrow[r, "\mathbb{1}_X\square m"] & X\square X \arrow[d, "m"] \\
X\square X \arrow[rr, "m"'] & & X
\end{tikzcd}$$

$$\begin{tikzcd}
\{*\}\square X \arrow[r, "e\square\mathbb{1}_X"] \arrow[dr, "\lambda^{\mathbf{GrdSet}}_X"'] & X\square X \arrow[d, "m"] & X\square\{*\} \arrow[l, "\mathbb{1}_X\square e"'] \arrow[dl, "\rho^{\mathbf{GrdSet}}_X"] \\
& X &
\end{tikzcd}$$

Three Induced Functors

The Change of Base Functor

Given a set map $f : A \to B$ there is an induced functor $f^* : \mathbf{Set}/B \to \mathbf{Set}/A$ between slice categories called a **change of base** functor:

- It takes a set map $\chi : X \to B$ and returns the pullback map $f^*(\chi) : X_\chi \times_f A \to A$ of χ along f

$$\begin{tikzcd} A \arrow[d, "f"'] \\ B \end{tikzcd} \qquad\qquad \begin{tikzcd} \mathbf{Set}/A \arrow[d, shift right=8, "\Sigma_f"'] \arrow[d, shift left=8, "\Pi_f"] \\ \mathbf{Set}/B \arrow[u, "f*"] \end{tikzcd}$$

The Left Adjoint

$$\begin{tikzcd} A \arrow[d, "f"'] \\ B \end{tikzcd} \qquad \begin{tikzcd} \mathbf{Set}/A \arrow[d, shift right=6, "\Sigma_f"'] \arrow[d, shift left=6, "\Pi_f"] \\ \mathbf{Set}/B \arrow[u, "f*"] \end{tikzcd}$$

The Dependent Sum $\Sigma_f : \mathbf{Set}/A \to \mathbf{Set}/B$

On objects the left adjoint is simply composing an object of $\mathbf{Set}/A$ with f resulting in an object in $\mathbf{Set}/B$.

The Right Adjoint

$$
\begin{tikzcd}
A \arrow[d, "f"'] \\
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\qquad\qquad
\begin{tikzcd}
\mathbf{Set}/A \arrow[d, shift right=6, "\Sigma_f"'] \arrow[d, shift left=6, "\Pi_f"] \\
\mathbf{Set}/B \arrow[u, "f*"]
\end{tikzcd}
$$

The Dependent Product $\Pi_f : \mathbf{Set}/A \to \mathbf{Set}/B$

Let $\psi : Y \to A$ be any morphism in $\mathbf{Set}/A$. The map $\Pi_f(\psi) : \Gamma \to B$ is constructed by specifying the fiber Γ_b over each point $b \in B$:

- $\forall b \in B$ consider its fiber A_b along the map f
- $\forall c \in A_b$ consider its fiber Y_c along the map ψ
- The fiber Γ_b along $\Pi_f(\psi)$ is the product $\displaystyle\prod_{c \in A_b} Y_c$

Following this construction $\forall b \in B$ gives the complete map $\Gamma := \displaystyle\coprod_{b \in B} \Gamma_b \to B$.

Internal Hom in **GrdSet**

$-\Box B$: **GrdSet** $\to$ **GrdSet**

Consider $b : B \to \mathbb{N}$

$$-\Box B = \Sigma_{\mu_{\{*\}}} \Sigma_{T(b)} T(!_B)^*$$

$$\begin{tikzcd}
A\Box B \arrow[r, "T(!_B)^*(a)"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & T(B) \arrow[r, "T(b)"] \arrow[d, "T(!_B)"] & T^2(\{*\}) \arrow[r, "\mu_{\{*\}}"] & T(\{*\}) \\
A \arrow[r, "a"] & T(\{*\}) & &
\end{tikzcd}$$

Internal Hom in **GrdSet** cont.

$$\begin{tikzcd} B \arrow[d, "b"'] \\ \mathbb{N} \end{tikzcd} \qquad \begin{tikzcd} \mathbf{Set}/B \arrow[d, shift right=6, "\Sigma_b"'] \arrow[d, shift left=6, "\Pi_b"] \\ \mathbf{Set}/\mathbb{N} \arrow[u, "b*"'] \end{tikzcd}$$

$[B, -]$: **GrdSet** $\to$ **GrdSet**

The adjoint is found by taking the right adjoint of each factor in the composition and reversing the order in which they are composed:

$$-\Box B = \Sigma_{\mu_{\{*\}}} \Sigma_{T(b)} T(!_B)^*$$

$$[B, -] = \Pi_{T(!_B)} T(b)^* \mu^*_{\{*\}}$$

Internal Hom in **GrdSet** cont.

How does $T(b)^*\mu^*_{\{*\}}$ act on $a : A \to \mathbb{N}$

$$
\begin{tikzcd}
(A_a \times_{\mu^*_{\{*\}}(a)} T(\mathbb{N}))_{\mu^*_{\{*\}}(a)} \times_{T(b)} T(B) \arrow[r, "T(b)^*\mu^*_{\{*\}}(a)"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & T(B) \arrow[d, "T(b)"] \\
A_a \times_{\mu^*_{\{*\}}(a)} T(\mathbb{N}) \arrow[r, "\mu^*_{\{*\}}(a)"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & T(\mathbb{N}) \arrow[d, "\mu_{\{*\}}"] \\
A \arrow[r, "a"] & \mathbb{N}
\end{tikzcd}
$$

Internal Hom in **GrdSet** cont.

$(A_a \times_{\mu^*_{\{*\}}(a)} T(\mathbb{N}))_{\mu^*_{\{*\}}(a)} \times_{T(b)} T(B) \to T(B)$

A triple $((\alpha, t), \beta)$, an element of $\alpha \in A$ whose arity is factored via the string $t \in T(\mathbb{N})$, together with a word $\beta \in T(B)$ whose letters' arities agree with t

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$\Pi_{T(!_B)} T(b)^* \mu^*_{\{*\}}(a) : \Gamma \to \mathbb{N}$

- $\forall n \in \mathbb{N}$ consider its fiber $T(B)_n$ along the map $T(!_B) : T(B) \to \mathbb{N}$
- $\forall \beta \in T(B)_n$ has a fiber $((A_a \times_{\mu^*_{\{*\}}} T(\mathbb{N}))_{\mu^*_{\{*\}}(a)} \times_{T(b)} T(B))_\beta$ along the map $T(b)^* \mu^*_{\{*\}}(a)$

$$\Gamma_n := \prod_{\beta \in T(B)_n} ((A_a \times_{\mu^*_{\{*\}}} T(\mathbb{N}))_{\mu^*_{\{*\}}(a)} \times_{T(b)} T(B))_\beta$$

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In Γ_n, every word $\beta \in T(B)$ of length n is paired with an element of A whose arity factors to agree with the sum of the arities of the letters in β

Internal Hom in **GrdSet** cont.

We shall denote $\Pi_{T(!_B)} T(b)^* \mu^*_{\{*\}}(a) : \Gamma \to \mathbb{N}$ more suggestively as

$H_{B,A} : [B, A] \to \mathbb{N}$

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For $\beta \in T(B)_n$ such that $\beta = \beta_1 \beta_2 ... \beta_n$ with $\beta_i \in B$

$$[B, A]_n = \coprod_{\beta \in T(B)_n} \{((\alpha, t), \beta) |$$

$$\alpha \in A, t \in T(\mathbb{N})_n, a(\alpha) = \sum_i t_i, T(b)(\beta) = t\}$$

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$$\cong \prod_{\beta \in T(B)_n} \{(\alpha, \beta) | \alpha \in A, a(\alpha) = \sum_i b(\beta_i)\}$$

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$$\cong \prod_{\beta \in T(B)_n} \{(\alpha, \beta)|\alpha \in A, a(\alpha) = \sum_i b(\beta_i)\}$$

A 'map' $\gamma \in [B, A]_n$ may be thought of as a choice of elements from A to correspond to each string of n elements from B in such a way as to preserve arities.

Degenerate Graded Sets

The tautological operad should be generated by a set to give classical algebras.

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The tautological operad should be generated by a set to give classical algebras.

- But any set may be seen as a 'degenerate' graded set in a canonical way.

Definition

A **degenerate graded set** is a graded set concentrated in degree 0. More precisely, it is a graded set $x : X \to \mathbb{N}$ whose arity map factors as $x = [0] \circ !_X : X \to \{*\} \to \mathbb{N}$ in which $[0] : \{*\} \to \mathbb{N}$ is the 'name of zero' map which identifies the element $0 \in \mathbb{N}$.

The Internal Hom $H_{X,X} : [X, X] \to \mathbb{N}$

Let $x : X \to \mathbb{N}$ be a degenerate graded set.

$H_{X,X} : [X, X] \to \mathbb{N}$

For $\xi \in T(X)_n$ such that $\xi = \xi_1 \xi_2 ... \xi_n$ with $\xi_i \in X$

$$[X, X]_n = \prod_{\xi \in T(X)_n} \{((\chi, t), \xi) | \chi \in X, t \in T(\mathbb{N})_n, T(x)(\xi) = t,$$

$$x(\chi) = 0 = \sum_i t_i = \sum_i 0\}$$

$$\cong \prod_{\xi \in T(X)_n} \{(\chi, \xi) | \chi \in X\} \cong \prod_{\xi \in T(X)_n} \{\chi | \chi \in X\}$$

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$$\cong \prod_{\xi \in T(X)_n} \{(\chi, \xi) | \chi \in X\} \cong \prod_{\xi \in T(X)_n} \{\chi | \chi \in X\}$$

$$\cong \mathbf{Set}(X^n, X)$$

The Classical Tautological Operad $taut(X)$

We will need the following:

$$\epsilon^X : [X, -]\Box X \Rightarrow \mathbb{1}_{\mathbf{GrdSet}} \qquad \epsilon^X_X : [X, X]\Box X \to X$$

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$$\frac{\epsilon^X_X \circ (\mathbb{1}_{[X,X]}\Box\epsilon^X_X) : [X, X]\Box[X, X]\Box X \to X}{m : [X, X]\Box[X, X] \to [X, X]} \qquad \frac{\lambda_X : \{*\}\Box X \to X}{e : \{*\} \to [X, X]}$$

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Definition

The **tautological operad** on a degenerate graded set $x : X \to \mathbb{N}$ is the graded set $taut(X) : [X, X] \to \mathbb{N}$ equipped with the multiplication map $m : [X, X]\Box[X, X] \to [X, X]$ and unit map $e : \{*\} \to [X, X]$.

Algebra for a Graded Set

Definition

A **graded algebra for** a (non-symmetric) operad $o : O \to \mathbb{N}$ is a morphism of graded sets $f : O \to taut(X)$ for some degenerate graded set $x : X \to \mathbb{N}$.

Theorem

An algebra for an operad O is an algebra for every operad P which maps to O. In particular, an algebra for O is an algebra for every sub-operad of O.

$$P \xrightarrow{g} O \xrightarrow{f} taut(X) \qquad \tilde{O} \xhookrightarrow{i} O \xrightarrow{f} taut(X)$$

$$P \xrightarrow{\quad f(g) \quad} taut(X) \qquad \tilde{O} \xrightarrow{\quad f(i) \quad} taut(X)$$

Generalization

All of this generalizes naturally to setting of globular sets and collections.

Globular Sets

Definition

The **globe category** $\mathbb{G}$ has $\mathbb{N}$ as its set of objects. Its morphisms are generated by $\sigma_n : n \to n+1$ and $\tau_n : n \to n+1$ for all $n \in \mathbb{N}$ subject to the relations $\sigma_{n+1} \circ \sigma_n = \tau_{n+1} \circ \sigma_n$ and $\sigma_{n+1} \circ \tau_n = \tau_{n+1} \circ \tau_n$.

Definition

A *globular set* is a contravariant functor $G : \mathbb{G} \to \mathbf{Set}$. The category **Glob** of globular sets is the category of presheaves on $\mathbb{G}$.

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Definition

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Globular Sets

A globular set $G = (\{G_n\}_{n\in\mathbb{N}}, \{s_G^n\}, \{t_G^n\})$ consists of a family of sets $\{G_n\}_{n\in\mathbb{N}}$ together with source and target maps $s_G = \{s_G^n : G_n \to G_{n-1}\}$ and $t_G = \{t_G^n : G_n \to G_{n-1}\}$ subject to the relations $s_G^n \circ s_G^{n+1} = s_G^n \circ t_G^{n+1}$ and $t_G^n \circ s_G^{n+1} = t_G^n \circ t_G^{n+1}$ in each dimension $n \in \mathbb{N}$.

The Free Strict ω-category Monad $\mathcal{T}$: **Glob** $\rightarrow$ **Glob**

Definition

The monad $\mathcal{T}$: **Glob** $\rightarrow$ **Glob** takes a globular set X and returns the underlying globular set of the free strict ω-category generated by X. It extends globular set homomorphisms in the canonical way.

The Free Strict ω-category Monad $\mathcal{T} : \mathbf{Glob} \to \mathbf{Glob}$

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The monad $\mathcal{T} : \mathbf{Glob} \to \mathbf{Glob}$ takes a globular set X and returns the underlying globular set of the free strict ω-category generated by X. It extends globular set homomorphisms in the canonical way.

- Consider the globular set $\mathbf{1}$ with exactly one cell in every dimension.
  - We may think of the elements of $\mathcal{T}(\mathbf{1})$ as all possible globular pasting diagrams.

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- Consider the globular set $\mathbf{1}$ with exactly one cell in every dimension.
  - We may think of the elements of $\mathcal{T}(\mathbf{1})$ as all possible globular pasting diagrams.

Definition

A **collection** is a globular set X equipped with a globular set homomorphism $x : X \to \mathcal{T}(\mathbf{1})$ called the arity map.

- The category **Col** is the slice category $\mathbf{Glob}/\mathcal{T}(\mathbf{1})$
  - This replaces numeric arities with 'arity shapes'!

Monoidal Structure in **Col**

Definition

Let $x : X \to \mathcal{T}(\mathbf{1})$ and $y : Y \to \mathcal{T}(\mathbf{1})$ be a pair of collections. Their composition tensor product $x \Box y : X \Box Y \to \mathcal{T}(\mathbf{1})$ is defined by the diagram:

$$
\begin{tikzcd}
X \Box Y \arrow[r] \arrow[d] \arrow[dr, phantom, very near start, "\lrcorner"] & \mathcal{T}(Y) \arrow[r, "\mathcal{T}(y)"] \arrow[d, "\mathcal{T}(!_Y)"] & \mathcal{T}^2(\mathbf{1}) \arrow[r, "\mu_{\mathbf{1}}"] & \mathcal{T}(\mathbf{1}) \\
X \arrow[r, "x"] & \mathcal{T}(\mathbf{1}) & &
\end{tikzcd}
$$

where $!_Y : Y \to \mathbf{1}$ is the unique map from Y to the terminal globular set. The underlying globular set $X \Box Y$ is the pullback of x and $\mathcal{T}(!_Y)$ with the arity globular set map $x \Box y$ defined to be the composition along the top row.

Globular Operads

Unit for $\Box$

The monoidal unit for $\Box$ is the collection $I : \mathbf{1} \hookrightarrow \mathcal{T}(\mathbf{1})$

Definition

A **globular operad** is a monoid in **Col** with respect to the tensor product $\Box$.

Degenerate Collections

The collection $[id] : \mathbf{1} \to \mathcal{T}(\mathbf{1})$ identifies the special infinite chain of iterated identities created in $\mathcal{T}(\mathbf{1})$ when freely generating on $\mathbf{1}$

- Any globular set A can be seen as a 'degenerate' collection via the arity map:

$$[id] \circ !_A : A \to \mathbf{1} \to \mathcal{T}(\mathbf{1})$$

Three Induced Functors

The Change of Base Functor

Given a set map $\varphi : \mathcal{A} \to \mathcal{B}$ there is an induced functor $\varphi^* : \mathbf{Glob}/\mathcal{B} \to \mathbf{Glob}/\mathcal{A}$ between slice categories called a **change of base** functor:

- It takes a globular set map $\chi : \mathcal{X} \to \mathcal{B}$ and returns the pullback map $\varphi^*(\chi) : X_\chi \times_\varphi \mathcal{A} \to \mathcal{A}$ of χ along φ

$$\begin{tikzcd} \mathcal{A} \arrow[d, "\varphi"'] \\ \mathcal{B} \end{tikzcd} \qquad \begin{tikzcd} \mathbf{Glob}/\mathcal{A} \arrow[d, shift right=6, "\Sigma_\varphi"'] \arrow[d, shift left=6, "\Pi_\varphi"] \\ \mathbf{Glob}/\mathcal{B} \arrow[u, "\varphi*"'] \end{tikzcd}$$

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The Dependent Sum $\Sigma_\varphi : \mathbf{Glob}/\mathcal{A} \to \mathbf{Glob}/\mathcal{B}$

On objects the left adjoint is simply composing an object of $\mathbf{Glob}/\mathcal{A}$ with φ resulting in an object in $\mathbf{Glob}/\mathcal{B}$.

The Right Adjoint

$$\begin{tikzcd} \mathcal{A} \arrow[d, "\varphi"'] \\ \mathcal{B} \end{tikzcd} \qquad \begin{tikzcd} \mathbf{Glob}/\mathcal{A} \arrow[d, shift right=6, "\Sigma_\varphi"'] \arrow[d, shift left=6, "\Pi_\varphi"] \\ \mathbf{Glob}/\mathcal{B} \arrow[u, "\varphi*"] \end{tikzcd}$$

The Dependent Product $\Pi_\varphi : \mathbf{Glob}/\mathcal{A} \to \mathbf{Glob}/\mathcal{B}$

Let $\psi : \mathcal{Y} \to \mathcal{A}$ be any morphism in $\mathbf{Glob}/\mathcal{A}$. The map $\Pi_\varphi(\psi) : \Gamma \to \mathcal{B}$ is constructed by specifying the fiber over each point:

- $\forall b \in \mathcal{B}$ consider its fiber $\mathcal{A}_b$ along the map φ
- $\forall c \in \mathcal{A}_b$ consider its fiber $\mathcal{Y}_c$ along the map ψ
- The fiber Γ_b along $\Pi_\varphi(\psi)$ is the product $\displaystyle\prod_{c \in \mathcal{A}_b} \mathcal{Y}_c$

Following this construction $\forall b \in B$ gives the complete map $\Gamma := \displaystyle\coprod_{b \in \mathcal{B}} \Gamma_b \to \mathcal{B}$.

Internal Hom in **Col**

$-\Box\mathcal{B}$: **Col** $\to$ **Col**

Consider $b : \mathcal{B} \to \mathcal{T}(\mathbf{1})$

$$-\Box\mathcal{B} = \Sigma_{\mu_{\mathbf{1}}}\Sigma_{\mathcal{T}(b)}\mathcal{T}(!_{\mathcal{B}})^*$$

$$\begin{tikzcd}
\mathcal{A}\Box\mathcal{B} \arrow[r, "\mathcal{T}(!_{\mathcal{B}})^*(a)"] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & T(\mathcal{B}) \arrow[r, "\mathcal{T}(b)"] \arrow[d, "\mathcal{T}(!_{\mathcal{B}})"] & \mathcal{T}^2(\mathbf{1}) \arrow[r, "\mu_{\mathbf{1}}"] & \mathcal{T}(\mathbf{1}) \\
\mathcal{A} \arrow[r, "a"] & \mathcal{T}(\mathbf{1}) & &
\end{tikzcd}$$

Internal Hom in **Col** cont.

$$\begin{tikzcd} \mathcal{B} \arrow[d, "b"'] \\ \mathcal{T}(\mathbf{1}) \end{tikzcd} \qquad \begin{tikzcd} \mathbf{Glob}/\mathcal{B} \arrow[d, shift right=8, "\Sigma_b"'] \arrow[d, shift left=8, "\Pi_b"] \\ \mathbf{Glob}/\mathcal{T}(\mathbf{1}) \arrow[u, "b*"] \end{tikzcd}$$

$[\mathcal{B}, -]$: **Col** $\to$ **Col**

The adjoint is found by taking the right adjoint of each factor in the composition and reversing the order in which they are composed:

$$-\Box\mathcal{B} = \Sigma_{\mu_{\mathbf{1}}}\Sigma_{\mathcal{T}(b)}\mathcal{T}(!_{\mathcal{B}})^*$$

$$[\mathcal{B}, -] = \Pi_{\mathcal{T}(!_{\mathcal{B}})}\mathcal{T}(b)^*\mu_{\mathbf{1}}^*$$

Internal Hom in **Col** cont.

How does $\mathcal{T}(b)^*\mu_{\mathbf{1}}^*$ act on $a : \mathcal{A} \to \mathcal{T}(\mathbf{1})$

$$
\begin{tikzcd}
(\mathcal{A}_a \times_{\mu_{\mathbf{1}}^*(a)} \mathcal{T}(\mathcal{T}(\mathbf{1})))_{\mu_{\mathbf{1}}^*(a)} \times_{\mathcal{T}(b)} \mathcal{T}(\mathcal{B}) \arrow[r, "\mathcal{T}(b)^*\mu_{\mathbf{1}}^*(a)"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{T}(\mathcal{B}) \arrow[d, "\mathcal{T}(b)"] \\
\mathcal{A}_a \times_{\mu_{\mathbf{1}}^*(a)} \mathcal{T}(\mathcal{T}(\mathbf{1})) \arrow[r, "\mu_{\mathbf{1}}^*(a)"] \arrow[d, "\pi_1"'] \arrow[dr, phantom, "\lrcorner", very near start] & \mathcal{T}(\mathcal{T}(\mathbf{1})) \arrow[d, "\mu_{\mathbf{1}}"] \\
\mathcal{A} \arrow[r, "a"] & \mathcal{T}(\mathbf{1})
\end{tikzcd}
$$

Internal Hom in **Col** cont.

$(\mathcal{A}_a \times_{\mu_{\mathbf{1}}^*(a)} \mathcal{T}(\mathcal{T}(\mathbf{1})))_{\mu_{\mathbf{1}}^*(a)} \times_{\mathcal{T}(b)} \mathcal{T}(\mathcal{B}) \to \mathcal{T}(\mathcal{B})$

Once again, this map is simply second projection.

We can understand its dependent product with respect to $\mathcal{T}(!_{\mathcal{B}})$ by analogy to the graded set construction.

In **GrdSet**

We associated to a string of n elements in B, an element of A whose arity was the sum of the arities of the elements in B.

Internal Hom in **Col** cont.

$(\mathcal{A}_a \times_{\mu_{\mathbf{1}}^*(a)} \mathcal{T}(\mathcal{T}(\mathbf{1})))_{\mu_{\mathbf{1}}^*(a)} \times_{\mathcal{T}(b)} \mathcal{T}(\mathcal{B}) \to \mathcal{T}(\mathcal{B})$

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We associated to a string of n elements in B, an element of A whose arity was the sum of the arities of the elements in B.

$\Pi_{\mathcal{T}(!_{\mathcal{B}})} \mathcal{T}(b)^* \mu_{\mathbf{1}}^*(a) : \Gamma \to \mathcal{T}(\mathbf{1})$

Each σ-fiber consists of all the possible ways to associate to each word $\beta \in \mathcal{T}(\mathcal{B})$, glued together per the pasting scheme σ, a k-cell of $\mathcal{A}$ whose arity shape is the composite of the arity shapes of cells in $\mathcal{B}$ which compose to form β.

Internal Hom in **Col** cont.

We shall denote $\Pi_{\mathcal{T}(!_{\mathcal{B}})}\mathcal{T}(b)^{*}\mu_{\mathbf{1}}^{*}(a) : \Gamma \to \mathcal{T}(\mathbf{1})$ more suggestively as

$$H_{\mathcal{B},\mathcal{A}} : [\mathcal{B},\mathcal{A}] \to \mathcal{T}(\mathbf{1})$$

The tautological operad should be generated by a globular set to give the appropriate notion of algebras.

Internal Hom in **Col** cont.

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The tautological operad should be generated by a globular set to give the appropriate notion of algebras.

- We again use a 'degenerate' globular set $x = [id] \circ !_{\mathcal{X}} : \mathcal{X} \to \mathbf{1} \to \mathcal{T}(\mathbf{1})$ to build the internal hom

$$H_{\mathcal{X},\mathcal{X}} : [\mathcal{X}, \mathcal{X}] \to \mathcal{T}(\mathbf{1})$$

Internal Hom in **Col** cont.

<u>Note</u>: The step in **GrdSet** at which the internal hom 'collapsed' when all arities where 0, in this case still remembers the dimensions of each cell, even if its underlying shape is in a certain sense 'empty'.

Internal Hom in **Col** cont.

Note: The step in **GrdSet** at which the internal hom 'collapsed' when all arities where 0, in this case still remembers the dimensions of each cell, even if its underlying shape is in a certain sense 'empty'.

$H_{X,X} : [X, X] \to \mathbb{N}$ in **GrdSet**

We associated to a string of n elements in X, an element of X whose arity was the sum of the arities of the n elements from X.

Internal Hom in **Col** cont.

<u>Note:</u> The step in **GrdSet** at which the internal hom ‘collapsed’ when all arities where 0, in this case still remembers the dimensions of each cell, even if its underlying shape is in a certain sense ‘empty’.

$H_{X,X} : [X, X] \to \mathbb{N}$ in **GrdSet**

We associated to a string of n elements in X, an element of X whose arity was the sum of the arities of the n elements from X.

$H_{\mathcal{X},\mathcal{X}} : [\mathcal{X}, \mathcal{X}] \to \mathcal{T}(\mathbf{1})$ in **Col**

Each σ-fiber consists of all the possible ways to associate to each word $\xi \in \mathcal{T}(\mathcal{X})$, glued together per the pasting scheme σ, a k-cell of $\mathcal{X}$ whose arity shape is the composite of the arity shapes of cells in $\mathcal{X}$ which compose to form ξ.

The Globular Tautological Operad $gtaut(X)$

$$\epsilon^{\mathcal{X}} : [\mathcal{X}, -]\Box\mathcal{X} \Rightarrow \mathbb{1}_{\mathbf{Col}} \qquad \epsilon^{\mathcal{X}}_{\mathcal{X}} : [\mathcal{X}, \mathcal{X}]\Box\mathcal{X} \to \mathcal{X}$$

$$\frac{\epsilon^{\mathcal{X}}_{\mathcal{X}} \circ (\mathbb{1}_{[\mathcal{X},\mathcal{X}]}\Box\epsilon^{\mathcal{X}}_{\mathcal{X}}) : [\mathcal{X}, \mathcal{X}]\Box[\mathcal{X}, \mathcal{X}]\Box\mathcal{X} \to \mathcal{X}}{m : [\mathcal{X}, \mathcal{X}]\Box[\mathcal{X}, \mathcal{X}] \to [\mathcal{X}, \mathcal{X}]} \qquad \frac{\lambda_{\mathcal{X}} : \mathbf{1}\Box\mathcal{X} \to \mathcal{X}}{e : \mathbf{1} \to [\mathcal{X}, \mathcal{X}]}$$

Definition

The **tautological operad** on a degenerate graded set $x : \mathcal{X} \to \mathcal{T}(\mathbf{1})$ is the collection $gtaut(\mathcal{X}) : [\mathcal{X}, \mathcal{X}] \to \mathcal{T}(\mathbf{1})$ equipped with the multiplication map $m : [\mathcal{X}, \mathcal{X}]\Box[\mathcal{X}, \mathcal{X}] \to [\mathcal{X}, \mathcal{X}]$ and unit map $e : \mathbf{1} \to [\mathcal{X}, \mathcal{X}]$.

Algebra for a Collection

Definition

An **algebra for** a globular operad $o : \mathcal{O} \to \mathcal{T}(\mathbf{1})$ is a morphism of graded sets $\varphi : \mathcal{O} \to gtaut(X)$ for some degenerate graded set $x : \mathcal{X} \to \mathcal{T}(\mathbf{1})$.

Theorem

An algebra for an operad $\mathcal{O}$ is an algebra for every operad $\mathcal{P}$ which maps to $\mathcal{O}$. In particular, an algebra for $\mathcal{O}$ is an algebra for every sub-operad of $\mathcal{O}$.

$$\mathcal{P} \xrightarrow{\psi} O \xrightarrow{\varphi} gtaut(X) \qquad \tilde{\mathcal{O}} \xhookrightarrow{i} \mathcal{O} \xrightarrow{\varphi} gtaut(\mathcal{X})$$

$$\mathcal{P} \xrightarrow{\varphi(\psi)} gtaut(\mathcal{X}) \qquad \tilde{\mathcal{O}} \xrightarrow{\varphi(i)} gtaut(\mathcal{X})$$

Thank You

References

- B. Fresse. *Modules over Operads and Functors*. Lectures Notes in Mathematics. Spring New York, 1967.
- P. Johnstone. *Sketches of an Elephant: A Topos Theory Compendium*. Oxford logic guides. Oxford University Press, 2002.
- P. Johnstone. *Topos Theory*. Dover Books on Mathematics. Dover Publications, 2014.
- T. Leinster. *General Operads and Multicategories*. London Mathematical Society Lecture Notes Series. Cambridge University Press, 2004.
- T. Leinster. *Higher Operads, Higher Categories*. London Mathematical Society Lecture Notes Series. Cambridge University Press, 2003.
- S. MacLane and I. Moerdijk. *Sheaves in Geometry and Logic: A First Indtroduction to Topos Theory*. Universitext. Springer New York, 1994.
- J. May *The Geometry of Iterated Loop Spaces*. Lecture Notes in Mathematics. Springer Berlin Heidelberg, 2006.